On the ill-posedness of observation problems

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Abstract. Observation problems are restricted here to problems of estimation of state variables (or more generally, internal variables) from two sources of information: online measurements of some variables and the dynamic model relating the quantities to be estimated and the measurements. In the control theory engineering literature the tremendous success of the Kalman filter has left little room to numerical analysis approaches to observation problems. This work is a contribution to the building of a tunnel between numerical analysis and control theory literature on observation problems. The first brick is the statement that state estimation is an ill-posed inverse problem. In the present communication attention is focused on linear systems (with constant or non constant coefficients) for which popular asymptotic estimators (Luenberger observer and Kalman observer) are shown to be regularizations of the ill-posed estimation problem.

Keywords. Observer design; State estimation; Ill-posed inverse problems

1 Extended abstract

For control systems

$$\left\{ \begin{array}{rcl} \dot{x} & = & f(u, x)\,, \\ y & = & h(x)\,, \end{array} \right. \tag{1}$$

state estimation consists of the *online* estimation of $x(t)$ from the knowledge of f, h and time histories $([t_0, t] \ni \tau \longmapsto u(\tau), y(\tau))$ of u and y. Here $\dot{x}$ stands for derivative of x with respect to t, t_0 is an initial instant, and f and h are sufficiently smooth functions of their arguments respectively defined on regions of $\mathbb{R}^m \times \mathbb{R}^n$, and $\mathbb{R}^n$, and respectively ranging in $\mathbb{R}^n$ and $\mathbb{R}^p$, where n, m and p natural integers. This problem is central in systems theory and is under investigation since the pioneering work of R. E. Kalman in the late fifties addressing its linear context. A complete nonlinear answer is still lacking. A general approach consists of a two part theory: one of *observability*, that is, derivation of conditions on f and h guaranteeing the ability to some how estimate x from the supposedly known data, and another part of the theory, the *observer design*, yielding algorithms for such an estimation of x.

Observability theory for system 1 has been extensively studied in [6, 4] by viewing the observation problem as the *invertibility* of the mapping

$$\ell_u \;:\; \; x(t_0) = x_0 \longmapsto y$$

for fixed values for u. In this context, the function f is assumed to be regular enough to yield unique solutions in $[t_0, \infty[$ to the differential equation in (1) given u and x_0.

This *state space* viewpoint of observability has been challenged in the early nineties by an algebraic approach [2] which may be qualified as a *trajectory* viewpoint. It amounts to replacing the previous map ℓ_u by the following one

$$\ell_t \;:\; \begin{pmatrix} u \\ x \end{pmatrix} \longmapsto \begin{pmatrix} u \\ y \end{pmatrix}$$

indexed by $t > t_0$, and where u, x and y stand for the corresponding functions restricted to $[t_0, t]$.

State estimation then reads as inversion of the maps l_t. This inverse problem then is seen to be an ill-posed posed one in the usual sense in inverse problems literature [3].

The paper is devoted to the clarification of this matter of fact. The linear case is first considered, building upon previous works [7, 1]. The first one shows in the context of constant coefficients linear systems how standard asymptotic observers may be seen as regularization operators. In the context of time-varying linear systems *dynamic inversion* [5] is invoked to show that well-known Kalman observer is a regularization of the ill-posed estimation problem.

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & -a_1 \, x_2 \, , \\ \dot{x}_2 & = & x_1 - a_2 \, x_2 \, , \\ y & = & x_2 \, . \end{array} \right.$$

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